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WHITNEY'S EXTENSION PROBLEM FOR MULTIVARIATE $C^{1,\omega}$-FUNCTIONS

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Dedicated to the memory of Evsey Dyn'kin

ABSTRACT. We prove that the trace of the space $C^{1,\omega}(\mathbb{R}^n)$ to an arbitrary closed subset $X \subset \mathbb{R}^n$ is characterized by the following "finiteness" property. A function $f : X \to \mathbb{R}$ belongs to the trace space if and only if the restriction $f|_Y$ to an arbitrary subset $Y \subset X$ consisting of at most $3\cdot 2^{n-1}$ can be extended to a function $f_Y \in C^{1,\omega}(\mathbb{R}^n)$ such that

$$\sup\{\|f_Y\|_{C^{1,\omega}} : \ Y \subset X, \ \operatorname{card} Y \leq 3 \cdot 2^{n-1}\} < \infty.$$

The constant $3 \cdot 2^{n-1}$ is sharp.

The proof is based on a Lipschitz selection result which is interesting in its own right.

1. MAIN RESULTS

The results of the paper are concerned with the following problem having its origin in two classical papers of Hassler Whitney [W1], [W2] which appeared in 1934.

Let $C^k(\mathbb{R}^n)$ be the space of k-times continuously differentiable functions f satisfying

$$\|f\|_{C^k(\mathbb{R}^n)} := \sum_{|\alpha| \leq k} \sup_{x \in \mathbb{R}^n} |D^\alpha f(x)| < \infty.$$

Here the sign " := " means "by definition".

Main Problem. What is a necessary and sufficient condition for a given function f defined on a set $X \subset \mathbb{R}^n$ to be the restriction to X of a function from $C^k(\mathbb{R}^n)$?

In other words, we are looking for a constructive description of the trace space $C^k(\mathbb{R}^n)|_X := \{F|_X : \ F \in C^k(\mathbb{R}^n)\}$. Here and below X denotes an arbitrary closed subset of $\mathbb{R}^n$, and $F|_X$ stands for a restriction to X of a function F defined on $\mathbb{R}^n$.

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Recall also that the trace space $A|_X := \{f|_X : \ f \in A\}$ of a (semi-)normed space A of functions $f : \mathbb{R}^n \to \mathbb{R}$ is equipped with the (Banach) trace (semi-)norm

$$\|f\|_{A|_X} := \inf\{\|F\|_A : \ F \in A, \ F|_X = f\}.$$

In [W1] Whitney gave a complete solution to the apparently easier analogue of the Main Problem for the space $J^k(\mathbb{R}^n)$ of k-jets, i.e., tuples $\{f_\alpha : |\alpha| \le k\}$ generated by C^k-functions. (Here $\{f_\alpha\} \in J^k(\mathbb{R}^n)$ if $f_\alpha = D^\alpha f, \ |\alpha| \le k$, for some $f \in C^k(\mathbb{R}^n)$.) He also proved the existence of a linear bounded extension operator from $J^k(\mathbb{R}^n)|_X$ into $C^k(\mathbb{R}^n)$.

In [W2] Whitney went on to consider the Main Problem itself. He solved it for the case $n = 1$; see Theorem A below. Apparently, the number "I" appearing in the title of [W2] suggests that he intended to subsequently also consider the *multidimensional* version of the problem. However, no further publications in this direction have appeared in the nearly sixty years which have passed since then.

On the other hand, in 1958 Glaeser [G] (see also [St], Ch. 6) proved an analogue of the first mentioned Whitney result for the space $J^{k,\omega}(\mathbb{R}^n)$ of k-jets generated by $C^{k,\omega}$-functions. Recall that the space $C^{k,\omega}(\mathbb{R}^n)$ is the subspace of $C^k(\mathbb{R}^n)$ defined by the norm

$$\|f\|_{C^{k,\omega}(\mathbb{R}^n)} := \|f\|_{C^k(\mathbb{R}^n)} \ + \sum_{|\alpha|=k} \sup_{x,y\in\mathbb{R}^n} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{\omega(\|x-y\|)}. \tag{1.1}$$

Here $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is a concave continuous function satisfying $\omega(+0) = 0$ while $\|x\| := \max\limits_{1\le i\le n} |x_i|$.

Let us note that it suffices to solve the Main Problem for the trace space $C^k(\mathbb{R}^n)|_Q$ with an arbitrary n-cube Q. The latter space, in turn, coincides with the union $\bigcup\limits_\omega C^{k,\omega}(\mathbb{R}^n)|_Q$, so the crucial step consists in a solution to the similar problem for the space $C^{k,\omega}(\mathbb{R}^n)$.

In this paper we give a solution to the Main Problem for the space $C^{1,\omega}(\mathbb{R}^n)$ (the results were announced in [BS1]).

In order to indicate the difficulties of the multidimensional situation and to give a motivation for our approach it will be useful to formulate a variant of Whitney's result in [W2].

Theorem A (essentially Whitney [W2]). *A bounded function $f : X \to \mathbb{R}$ belongs to $C^{k,\omega}(\mathbb{R})|_X$ if and only if*

$$\sup_Y \frac{|f[Y]| \operatorname{diam}(Y)}{\omega(\operatorname{diam}(Y))} \ < \ \infty.$$

Here the supremum is taken over all subsets Y of X consisting of $k+2$ points and $f[Y]$ denotes the divided difference of f with respect to the points of Y.

Remark 1.1. The original result of Whitney states that a bounded function $f \in C^k(\mathbb{R})|_X$ iff for all $x \in X$

$$\lim_{Y\to x} f[Y] \operatorname{diam}(Y) = 0$$

where $Y \subset X$ is taken as above. This result is a simple consequence of Theorem A.

The obvious obstacle to generalizing Theorem A to the multidimensional case is the absence of a natural notion of divided difference for multivariate functions.

But fortunately Theorem A can be reformulated in a way which eliminates this obstacle:

Theorem A′. *A function $f : X \to \mathbb{R}$ belongs to $C^{k,\omega}(\mathbb{R})|_X$ if and only if for every subset $Y \subset X$* consisting of *$k+2$-points, there exists a function $f_Y \in C^{k,\omega}(\mathbb{R})$ which interpolates f on Y and such that*

$$\sup_Y \|f_Y\|_{C^{k,\omega}(\mathbb{R})} \ < \ \infty.$$

Let us note that in this case we can use Lagrange polynomials to construct f_Y. The relationship between the Lagrange interpolation and divided differences immediately implies equivalence of these two results. Note also that Theorem A′ is untrue if $k+2$ is replaced by a smaller number.

The preceding theorem leads to the following:

Definition 1.2. The trace space $A|_X$ has the *finiteness property* if there exists a positive integer N such that the following holds:

A function $f : X \to \mathbb{R}$ belongs to $A|_X$ if there is a *bounded* in A family of functions $\{f_Y : Y \subset X,\ \mathrm{card}(Y) \le N\}$ such that f_Y interpolates f on Y.

We let $N_X[A]$ denote the smallest integer N having this property. Put

$$N[A] := \sup_X N_X[A]$$

where the supremum is taken over all closed subsets $X \subset \mathbb{R}^n$.

Thus, Theorem A′ can be stated in the form

$$N[C^{k,\omega}(\mathbb{R})] \ = \ k+2 \ ;$$

in particular, $N[C^{1,\omega}(\mathbb{R})] = 3$. We show that in the multidimensional case the number of points has, rather surprisingly, exponential growth with respect to the dimension.

Theorem 1.3 (finiteness). $N[C^{1,\omega}(\mathbb{R}^n)] = 3 \cdot 2^{n-1}$.

Moreover, the trace norm of a function $f \in C^{1,\omega}(\mathbb{R}^n)|_X$ is equivalent to

$$\sup_Y \{\|f_Y\|_{C^{1,\omega}(\mathbb{R}^n)|_Y} :\ Y \subset X,\ \mathrm{card}(Y) \le 3 \cdot 2^{n-1}\}$$

up to constants depending only on n.

Thus the result of the theorem is equivalent to the following assertions:

I. Suppose that the restriction of a function $f : X \to \mathbb{R}$ to every subset $Y \subset X$ consisting of at most $3 \cdot 2^{n-1}$ points can be extended to a function $F_Y \in C^{1,\omega}(\mathbb{R}^n)$ with $\|F_Y\|_{C^{1,\omega}(\mathbb{R}^n)} \le 1$. Then the function f itself can be extended to a function $F \in C^{1,\omega}(\mathbb{R}^n)$ with the norm $\|F\|_{C^{1,\omega}(\mathbb{R}^n)} \le \gamma(n)$.

II. There are a set $\tilde{X} \subset \mathbb{R}^n$ and a function $\tilde{f} : \tilde{X} \to \mathbb{R}$ such that the restriction $\tilde{f}|_Y$ to every set $Y \subset \tilde{X}$ with $\mathrm{card}(Y) \le 3 \cdot 2^{n-1} - 1$ can be extended to a function in the unit ball of $C^{1,\omega}(\mathbb{R}^n)$, but $\tilde{f} \notin C^{1,\omega}(\mathbb{R}^n)|_{\tilde{X}}$.

The main geometric tool of our proof is a Lipschitz selection theorem. For its formulation we let $\mathcal{M}$ denote a metric space with a metric ρ. Define the Lipschitz space $\mathrm{Lip}(\mathcal{M}, \mathbb{R}^n)$ of functions $f : \mathcal{M} \to \mathbb{R}^n$ by the seminorm

$$|f|_{\mathrm{Lip}(\mathcal{M},\mathbb{R}^n)} := \inf\{\lambda : \|f(x) - f(y)\| \le \lambda\rho(x, y),\ x, y \in \mathcal{M}\}\,. \tag{1.2}$$

Now let $F : \mathcal{M} \to \mathcal{A}_k(\mathbb{R}^n)$ be a set-valued mapping from $\mathcal{M}$ into the family $\mathcal{A}_k(\mathbb{R}^n)$ of all affine subspace in $\mathbb{R}^n$ of dimension $\le k$. We are looking for conditions on F under which F has a *Lipschitz selection*, that is, a function $f \in \mathrm{Lip}(\mathcal{M}, \mathbb{R}^n)$ satisfying $f(m) \in F(m)$ for every $m \in \mathcal{M}$.

To motivate our approach it is useful to note that the main result formulated below also holds for a *pseudometric* space $\mathcal{M}$, i.e., in this case $\rho(m, m')$ may be 0 for $m \ne m'$ and ρ may admit the value $+\infty$. Consider, in particular, $\rho \equiv 0$. Then $\mathrm{Lip}(\mathcal{M}, \mathbb{R}^n)$ consists of constants only, and therefore in this case we are looking for conditions under which $\bigcap_{m \in \mathcal{M}} F(m) \ne \emptyset$. The corresponding Helly type result (so-called the Sylvester theorem) states that this intersection is non-empty if $\bigcap_{m \in \tilde{\mathcal{M}}} F(m) \ne \emptyset$ for every subset $\tilde{\mathcal{M}} \subset \mathcal{M}$ consisting of at most $k+2$ points. The answer in the general case strikingly differs from the classical situation. In fact, the following result, which gives a partial answer to a problem formulated by the first named author (see [BS1]) holds.

Theorem 1.4. *Let $F : \mathcal{M} \to \mathcal{A}_k(\mathbb{R}^n)$ be a set-valued mapping such that for every subset $\tilde{\mathcal{M}} \subset \mathcal{M}$ consisting of at most 2^{k+1} points, the restriction $F|_{\tilde{\mathcal{M}}}$ has a Lipschitz selection $f_{\tilde{\mathcal{M}}}$ with $|f_{\tilde{\mathcal{M}}}|_{\mathrm{Lip}(\tilde{\mathcal{M}}, \mathbb{R}^n)} \le 1$. Then F has a Lipschitz selection $f \in F$ with the seminorm bounded by a constant depending only on n.*

The number 2^{k+1} in general cannot be diminished.

This result was first proved in [Sh1]; see also [Sh2]. In fact, the main Theorem 1.3 requires a generalization of Theorem 1.4 related to selection of set-valued mappings defined on metric graphs. We present the proof of this result, Theorem 2.3, in the next section.

Remark 1.5. The proof of Theorem 1.3 contains implicitly a Whitney type criterion for belonging of a function to the trace space. For instance, for the case of two variables ($X \subset \mathbb{R}^2$) the analog of Theorem A states that a function $f \in \dot{C}^{1,\omega}(\mathbb{R}^2)|_X$ if and only if $\max\{w_1(X), w_2(X)\}$ is finite.[1] Here $w_1(X)$ is the one-dimensional *Whitney number* defined as in Theorem A:

$$w_1(X) := \sup_Z \frac{|f[Z]| \operatorname{diam}(Z)}{\rho_1(Z)}$$

where the supremum is taken over all subsets $Z \subset X$ with $\operatorname{card} Z = 3$ lying on a line and $\rho_1(Z) := \omega(\operatorname{diam}(Z))$. The two-dimensional Whitney number $w_2(X)$ is defined by the formula

$$w_2(X) := \sup_{Z_1, Z_2} \frac{\|\nabla P_{Z_1} - \nabla P_{Z_2}\|}{\rho_2(Z_1, Z_2)}$$

where Z_1, Z_2 run over all pairs of non-degenerate triangles with vertices in X while ∇P_Z stands for the gradient of the affine polynomial interpolating f at the vertices of Z. In turn, the function ρ_2 is determined by

$$\rho_2(Z_1, Z_2) := \frac{\rho_1(Z_1)}{|\sin \theta(Z_1)|} + \frac{\rho_1(Z_2)}{|\sin \theta(Z_2)|} + \rho_1(Z_1 \cup Z_2)$$

where $\theta(Z)$ is the biggest angle of the triangle Z.

[1] $\dot{C}^{1,\omega}(\mathbb{R}^n)$ denotes the homogeneous space $C^{1,\omega}(\mathbb{R}^n)$. The seminorm in this space is defined by the second term in the right-hand side of (1.1).

The general case could be treated in the same way but with a set of Whitney's numbers which is fast increasing along with n. Therefore, the corresponding formulae will become more and more complicated. For instance, for $n = 3$ this set contains 4 numbers and ρ_4 is defined on the set of quadruples $(Z_i)_{i=1}^4$. Here Z_i is a non-degenerate triangle with vertices in X such that the angle $\theta(Z_i, Z_{i+1})$ between the two-dimensional planes generated by Z_i and Z_{i+1} respectively is different from 0 and π, if $i = 1$ and $i = 3$. In this case

$$\rho_4(Z_1, Z_2) := \frac{\rho_2(Z_1, Z_2)}{|\sin\theta(Z_1, Z_2)|} + \frac{\rho_2(Z_3, Z_4)}{|\sin\theta(Z_3, Z_4)|} + \rho_1(\bigcup_{i=1}^{4} Z_i).$$

Comparing this expression with the preceding ones we can note a kind of a doubling procedure which reflects the corresponding algorithm situated in the core of the proof of Theorem 1.4 (and of its analytic counterpart - Theorem 1.3).

Thus, in spite of a seeming disorder, there is a rather rigid structure in the trace space $C^{1,\omega}(\mathbb{R}^n)|_X$ inherited from that of $C^{1,\omega}(\mathbb{R}^n)$.

2. The Lipschitz selection theorem

Let Γ be a graph with the sets of vertices V_Γ and edges E_Γ. We shall write $v_1 \leftrightarrow v_2$ for $v_1, v_2 \in V_\Gamma$ joined by an edge.

Let $w_\Gamma : E_\Gamma \to [0, +\infty]$ be a weight. Define a function $\rho_\Gamma : \Gamma \times \Gamma \to [0, +\infty]$ by the formula

$$\rho_\Gamma(v, v') := \inf_{\{e_k\}} \sum_k w_\Gamma(e_k) \tag{2.1}$$

where $\{e_k\}$ runs over all finite paths in the graph Γ joining v and v'. We set $\rho_\Gamma(v, v) := 0$ and $\rho_\Gamma(v, v') := +\infty$, if the set of the paths $\{e_k\}$ in (2.1) is empty.

Clearly, the function ρ_Γ satisfies the triangle inequality but may assign the value $+\infty$ and may be 0 for a pair (v, v') with $v \neq v'$. Thus ρ_Γ is a *pseudometric* and (V_Γ, ρ_Γ) is a *pseudometric space*. We let $\mathrm{Lip}(V_\Gamma, \mathbb{R}^n)$ denote the space of mappings $f : V_\Gamma \to \mathbb{R}^n$ defined by the seminorm

$$|f|_{\mathrm{Lip}(V_\Gamma, \mathbb{R}^n)} := \inf\{\lambda : \ \|f(v) - f(v')\| \le \lambda \rho_\Gamma(v, v'), \text{ for all } v, v' \in V_\Gamma\}.$$

Definition 2.1. A subset $V \subset V_\Gamma$ is said to be *admissible* if, being regarded as a subgraph of the graph Γ, it has no isolated vertices.

Example 2.2. Every pseudometric space $(\mathcal{M}, \rho)$ is generated by the (complete) weighted graph $\Gamma = \Gamma(\mathcal{M})$ with $V_\Gamma := \mathcal{M}$, $E_\Gamma := \{(m, m') \in \mathcal{M} \times \mathcal{M} : \ m \neq m'\}$ and $w_\Gamma(m, m') := \rho(m, m')$. The triangle inequality for ρ implies $\rho_\Gamma = \rho$ in this case. Note that all subsets of $\Gamma(\mathcal{M})$ are admissible.

In view of this example the following result is a generalization of Lipschitz selection Theorem 1.4.

Theorem 2.3. *Let $F : V_\Gamma \to \mathcal{A}_k(\mathbb{R}^n)$ be a set-valued mapping. Assume that $F|_V$ has a Lipschitz selection f_V with*

$$|f_V|_{\mathrm{Lip}(V, \mathbb{R}^n)} \le 1 \tag{2.2}$$

for every admissible $V \subset V_\Gamma$ consisting of at most 2^{k+1} points. Then there is a Lipschitz selection f of F with

$$|f|_{\mathrm{Lip}(V_\Gamma, \mathbb{R}^n)} \le \gamma(k, n). \tag{2.3}$$

The number 2^{k+1} in general cannot be decreased.

Proof (*induction on* k). The result is trivial for $k = 0$. Suppose that the theorem holds for $0 \le k < n$ and prove it for $k + 1$.

Let $F : V_\Gamma \to \mathcal{A}_{k+1}(\mathbb{R}^n)$ satisfy (2.2) for every admissible $V \subset V_\Gamma$ consisting of at most 2^{k+2} points.

We will find the required Lipschitz selection in three steps. First we associate with (V_Γ, ρ_Γ) its "doubling" $(\tilde{V}_\Gamma, \tilde{\rho}_\Gamma)$ in the following way.

Let $\{v_1, v_2\} \subset V_\Gamma,\ v_1 \ne v_2$. Then $F|_{\{v_1,v_2\}}$ has a Lipschitz selection satisfying (2.2). In other words, there are points $x^i(v_1, v_2) \in F(v_i),\ i = 1, 2$, so that

$$\|x^1(v_1, v_2) - x^2(v_1, v_2)\| \le \rho_\Gamma(v_1, v_2). \tag{2.4}$$

Denote now by $Q_r(x)$ a (closed) cube (ℓ^n_∞-ball) with center at x and "radius" r. Note that it may coincide with $\mathbb{R}^n$ ($r = \infty$) or $\{x\}$ ($r = 0$). We write Q_r for $Q_r(0)$. Then by (2.4) the layer $F(v_2) + Q_{2\rho_\Gamma(v_1,v_2)}$ shifted by the vector $x(v_1, v_2) := x^1(v_1, v_2) - x^2(v_1, v_2)$ intersects the affine plane $F(v_1)$ by a non-empty convex subset, say $C(v_1, v_2)$, symmetric with respect to the point $x^1(v_1, v_2)$. Therefore, there is a family of layers $F(v_1) \cap \{L_i + Q_{r_i}\},\ i \in \mathcal{J}(v_1, v_2)$, satisfying the following conditions:

(a) L_i is an affine subset of $F(v_1)$ of dimension k passing through $x^1(v_1, v_2)$;
(b) $0 < r_i \le \infty$;
(c) $C(v_1, v_2)$ is intersection of this family of sets:

$$\begin{aligned} C(v_1, v_2) &:= F(v_1) \cap \{F(v_2) + Q_{2\rho_\Gamma(v_1,v_2)} + x(v_1, v_2)\} \\ &= \bigcap_{i\in\mathcal{J}(v_1,v_2)} (F(v_1) \cap \{L_i + Q_{r_i}\}). \end{aligned} \tag{2.5}$$

Note that r_i may assign the value $+\infty$ (e.g., if $F(v_1)$ is parallel to $F(v_2)$).

Now we define the promised pseudometric space $(\tilde{V}_\Gamma, \tilde{\rho}_\Gamma)$ by letting

$$\tilde{V}_\Gamma := \{\tilde{v} = (v_1, v_2, i) :\ v_1, v_2 \in V_\Gamma,\ v_1 \leftrightarrow v_2 \text{ and } i \in \mathcal{J}(v_1, v_2)\},$$

and for $\tilde{v} \ne \tilde{v}' := (v_1', v_2', i')$

$$\tilde{\rho}_\Gamma(\tilde{v}, \tilde{v}') := \rho_\Gamma(v_1, v_1') + r_i + r_{i'}.$$

Finally, we associate with F a set-valued mapping $\tilde{F} : \tilde{V}_\Gamma \to \mathcal{A}_k(\mathbb{R}^n)$ setting $\tilde{F}(\tilde{v}) := L_i$ for $\tilde{v} = (v_1, v_2, i) \in \tilde{V}_\Gamma$. Note that unlike F the mapping $\tilde{F}$ takes values of dimension $\le k$. Thus we can apply to $\tilde{F}$ the assertion of the k-th step of induction to establish

Proposition 2.4. *There is a Lipschitz selection $\tilde{f}$ of $\tilde{F}$ satisfying*

$$|\tilde{f}|_{\mathrm{Lip}(\tilde{V}_\Gamma)} \le \gamma(k, n). \tag{2.6}$$

Here and below $\mathrm{Lip}(\mathcal{M})$ *stands for* $\mathrm{Lip}(\mathcal{M}, \mathbb{R}^n)$.

Proof. By Example 2.2 we only have to prove that for every $\tilde{V} \subset \tilde{V}_\Gamma$ consisting of at most 2^{k+1} points there is a Lipschitz selection $\tilde{f}_{\tilde{V}}$ of the restriction $\tilde{F}|_{\tilde{V}}$ satisfying

$$|\tilde{f}_{\tilde{V}}|_{\mathrm{Lip}(\tilde{V})} \le 1. \tag{2.7}$$

For this goal we introduce a subset $V = V(\tilde{V})$ of V_Γ by

$$V := \mathrm{pr}_1(\tilde{V}) \cup \mathrm{pr}_2(\tilde{V}).$$

Here we set for $\tilde{v} = (v_1, v_2, i)$

$$\mathrm{pr}_k(\tilde{v}) := v_k,\ k = 1, 2,\ \mathrm{pr}_3(\tilde{v}) := i.$$

Sometimes we also write $v_k(\tilde{v}),\ i(\tilde{v})$ for $\mathrm{pr}_k(\tilde{v}),\ k = 1, 2$ and $\mathrm{pr}_3(\tilde{v})$ respectively.

Note that by the definition of $\tilde{V}_\Gamma$, for every $v \in V$ there is v' joining with v by an edge. Thus V is admissible and it consists of at most $2\,\mathrm{card}(\tilde{V}) \le 2^{k+2}$ points. Then by the condition on F there exists a Lipschitz selection $f_V : V \to \mathbb{R}^n$ satisfying (2.2). Verify now that for $v_1, v_2 \in V$

$$f_V(v_1) \in C(v_1, v_2); \tag{2.8}$$

see (2.5) for the definition. In fact, $f_V(v_1)$ belongs to $F(v_1)$ and

$$\begin{aligned} &\|f_V(v_1) - f_V(v_2) - x(v_1, v_2)\| \\ &\quad \le \|f_V(v_1) - f_V(v_2)\| + \|x^1(v_1, v_2) - x^2(v_1, v_2)\| \le 2\rho_\Gamma(v_1, v_2) \end{aligned}$$

(see (2.4)) so that $f_V(v_1)$ belongs to the layer $F(v_2) + Q_{2\rho_\Gamma(v_1,v_2)}$ shifted by $x(v_1, v_2)$, and (2.8) follows.

Define now the required Lipschitz selection $\tilde{f}_{\tilde{V}} : \tilde{V} \to \mathbb{R}^n$ by letting $\tilde{f}_{\tilde{V}}(\tilde{v})$ be a point of $\tilde{F}(\tilde{v})$ *nearest* to $f_V(v_1(\tilde{v}))$ (in the uniform metric $\|\cdot\|$). Then by (2.8) and (2.5) we have

$$f_V(v_1(\tilde{v})) \in L_{i(\tilde{v})} + Q_{r_{i(\tilde{v})}},$$

if $\tilde{v} \in \tilde{V}$. Since $\tilde{f}_{\tilde{V}}(\tilde{v})$ is a nearest point of $\tilde{F}(\tilde{v})$ to $f_V(v_1(\tilde{v}))$, we obtain

$$\|\tilde{f}_{\tilde{V}}(\tilde{v}) - f_V(v_1(\tilde{v}))\| \le r_{i(\tilde{v})}.$$

This leads to the inequality

$$\begin{aligned} \|\tilde{f}_{\tilde{V}}(\tilde{v}) - \tilde{f}_{\tilde{V}}(\tilde{v}')\| &\le r_{i(\tilde{v})} + r_{i(\tilde{v}')} + \|f_V(v_1(\tilde{v})) - f_V(v_1(\tilde{v}'))\| \\ &\le r_{i(\tilde{v})} + r_{i(\tilde{v}')} + \rho_\Gamma(v_1(\tilde{v}), v_1(\tilde{v}')). \end{aligned}$$

The right-hand side equals $\tilde{\rho}_\Gamma(\tilde{v}, \tilde{v}')$, that is, (2.7) holds. □

At the second step we determine a mapping $\hat{f} : \tilde{V}_\Gamma \to \mathbb{R}^n$ satisfying the following conditions:

(a) $\hat{f}(\tilde{v})$ depends on $\mathrm{pr}_1(\tilde{v})$ only; that is, $\hat{f}$, in fact, is defined on V_Γ;

(b) $|\hat{f}|_{\mathrm{Lip}(V_\Gamma)} \le \gamma$;

(c) $\hat{f}(\tilde{v})$ belongs to the $\gamma r_{i(\tilde{v})}$-neighborhood of $\tilde{f}(\tilde{v})$.

Here and below $\gamma = \gamma(k, n)$ is the constant from (2.6).

At the third step we define the desired Lipschitz selection $f : V_\Gamma \to \mathbb{R}^n$ of F setting

$$f(v) := \mathrm{Pr}(\hat{f}(v), F(v)) \tag{2.9}$$

where $\mathrm{Pr}(\cdot, L)$ stands for the orthogonal projection on an affine subset L.

We begin with the determination of $\hat{f}$. For this goal we introduce a pseudometric space $(\hat{V}_\Gamma, \hat{\rho}_\Gamma)$ setting $\hat{V}_\Gamma := \tilde{V}_\Gamma$ and

$$\hat{\rho}_\Gamma(\tilde{v}, \tilde{v}') := \gamma\rho_\Gamma(\mathrm{pr}_1(\tilde{v}), \mathrm{pr}_1(\tilde{v}')).$$

Using the Lipschitz selection $\tilde{f} : \tilde{V}_\Gamma \to \mathbb{R}^n$ of Proposition 2.4 we now define a set-valued mapping $\hat{F}$ from $\hat{V}_\Gamma$ into the family $\mathcal{K}(\mathbb{R}^n)$ of all cubes $Q_r(x)$ by setting

$$\hat{F}(\tilde{v}) := Q_r(\tilde{f}(\tilde{v})) \tag{2.10}$$

where $r := \gamma r_{i(\tilde{v})}$.

Proposition 2.5. *$\tilde{F}$ has a Lipschitz selection $\hat{f} : \hat{V}_\Gamma \to \mathbb{R}^n$ satisfying*

$$|\hat{f}|_{\mathrm{Lip}(\hat{V}_\Gamma)} \le 1. \tag{2.11}$$

Proof. By Proposition 2.4 the centers $\tilde{f}(\tilde{v}), \tilde{f}(\tilde{v}')$ of the cubes $\hat{F}(\tilde{v})$ and $\hat{F}(\tilde{v}')$ satisfy

$$\|\tilde{f}(\tilde{v}) - \tilde{f}(\tilde{v}')\| \le \gamma\tilde{\rho}_\Gamma(\tilde{v}, \tilde{v}') := \gamma(r_{i(\tilde{v})} + r_{i(\tilde{v}')} + \rho_\Gamma(v_1(\tilde{v}), v_1(\tilde{v}'))).$$

Since $\gamma r_{i(\tilde{v})}$ and $\gamma r_{i(\tilde{v}')}$ are "radii" of these cubes, there are points $x = x(\tilde{v}, \tilde{v}') \in \hat{F}(\tilde{v}),\ y = y(\tilde{v}, \tilde{v}') \in \hat{F}(\tilde{v}')$ so that

$$\|x - y\| \le \gamma\rho_\Gamma(v_1(\tilde{v}), v_1(\tilde{v}')) =: \hat{\rho}_\Gamma(\tilde{v}, \tilde{v}').$$

In other words, for every subset $\{\tilde{v}, \tilde{v}'\}$ of $\hat{V}_\Gamma$ consisting of two points the restriction $\hat{F}|_{\{\tilde{v},\tilde{v}'\}}$ has a Lipschitz selection $\hat{f}_{\{\tilde{v},\tilde{v}'\}}$ with $|\hat{f}_{\{\tilde{v},\tilde{v}'\}}|_{\mathrm{Lip}(\{\tilde{v},\tilde{v}'\})} \le 1$.

Thus to finish the proof it remains to use the following simple result.

Lemma 2.6. *Let K be a set-valued mapping from a pseudometric space $(\mathcal{M}, \rho)$ into $\mathcal{K}(\mathbb{R}^n)$. Suppose that for every subset $\mathcal{M}' \subset \mathcal{M}$ consisting of two points the restriction $K|_{\mathcal{M}'}$ has a Lipschitz selection $\kappa_{\mathcal{M}'} : \mathcal{M}' \to \mathbb{R}^n$ with $|\kappa_{\mathcal{M}'}|_{\mathrm{Lip}(\mathcal{M}')} \le 1$. Then K has a Lipschitz selection $\kappa : \mathcal{M} \to \mathbb{R}^n$ with $|\kappa|_{\mathrm{Lip}(\mathcal{M})} \le 1$.*

Proof. Projecting on the coordinate axes of $\mathbb{R}^n$ we reduce the proof to the case $n = 1$. So $K(m)$ for every $m \in \mathcal{M}$ is a segment $[a(m), b(m)]$ (which may coincide with a point or $\mathbb{R}$). Set $I_0 := [-1, 1]$ and define the desired selection by

$$\kappa(m) := \inf_{m' \in \mathcal{M}} \sup \{K(m') + \rho(m, m')I_0\}.$$

Thus we take the smallest among the right endpoints of the intervals $K(m') + \rho(m, m')I_0$. Clearly, $\kappa(m)$ does not exceed $b(m)$. On the other hand, $a(m) \le b(m') + \rho(m, m')$ for every $m' \in \mathcal{M}$, since there is a Lipschitz selection on the set $\{m, m'\}$ with the seminorm ≤ 1. Thus, $\kappa(m) \ge a(m)$, i.e., $\kappa(m) \in K(m)$.

We can prove that $|\kappa(m) - \kappa(m')| \le \rho(m, m')$ similarly. We leave the simple verification of this inequality to the reader. □

The proof of the proposition is complete. □

Note now that by Proposition 2.5

$$|\hat{f}(\tilde{v}) - \hat{f}(\tilde{v}')| \le \gamma\rho_\Gamma(\mathrm{pr}_1(\tilde{v}), \mathrm{pr}_1(\tilde{v}')) = 0,$$

if $\mathrm{pr}_1(\tilde{v}) = \mathrm{pr}_1(\tilde{v}')$. Thus $\hat{f}(\tilde{v})$ depends only on the first coordinate of $\tilde{v}$ and therefore defines a function on V_Γ (which we denote by the same symbol $\hat{f}$).

Thus it remains to prove that the selection $f \in F$ defined by (2.9) satisfies the Lipschitz condition (2.3). That is we have to show that for $v, v' \in V_\Gamma$

$$\|f(v) - f(v')\| \le \gamma_1\rho_\Gamma(v, v') \tag{2.12}$$

with a suitable constant $\gamma_1 = \gamma_1(k, n)$.

In view of definition (2.1) of ρ_Γ it suffices to prove (2.12) for the case of the vertices v, v' joined by an edge. So we assume that $v \leftrightarrow v'$ and prove (2.12) under this condition. For this goal we need two technical results.

Lemma 2.7. *Let $K_v := \bigcap Q_{r_{i(\tilde{v})}}(f(v))$ where the intersection is taken over all $\tilde{v} \in \tilde{V}_\Gamma$ with $\mathrm{pr}_1(\tilde{v}) = v$. Then for every $v \leftrightarrow v'$ the cube K_v satisfies*

$$K_v \cap F(v) \subset F(v') + Q_{2\bar{\gamma}\rho_\Gamma(v,v')} + x(v, v')$$

where $x(v, v') := x^1(v, v') - x^2(v, v')$ is defined in (2.4) and $\bar{\gamma} := (1 + \sqrt{n})\gamma$.

Proof. Note that $\tilde{v} := (v, v', i)$ where $i \in \mathcal{J}(v, v')$ (see (2.5)) belongs to $\tilde{V}_\Gamma$ by choice of v'. Thus (2.10) implies

$$\|\hat{f}(v) - \tilde{f}(\tilde{v})\| \le \gamma r_i \tag{2.13}$$

where $i = i(\tilde{v})$. Since $i \in \mathcal{J}(v, v')$, we also have $\tilde{f}(\tilde{v}) \in \tilde{F}(\tilde{v}) := L_i \subset F(v)$. So the definition of $f(v)$ as the orthogonal projection of $\hat{f}(v)$ on $F(v)$ (see (2.9)) yields

$$\|f(v) - \tilde{f}(\tilde{v})\| \le \sqrt{n}\|\hat{f}(v) - \tilde{f}(\tilde{v})\|.$$

Combining this inequality with (2.13) we have

$$\|f(v) - \tilde{f}(\tilde{v})\| \le \sqrt{n}\gamma r_{i(\tilde{v})}. \tag{2.14}$$

Let now $x \in K_v$. By the definition of K_v for every $i \in \mathcal{J}(v, v')(:= \{i(\tilde{v}) :\ \mathrm{pr}_1(\tilde{v}) = v,\ \mathrm{pr}_2(\tilde{v}) = v'\})$ we have $\|x - f(v)\| \le r_i$, which together with (2.14) implies

$$\|x - \tilde{f}(\tilde{v})\| \le (1 + \sqrt{n})r_i =: \bar{\gamma} r_i.$$

Since $\tilde{f}(\tilde{v}) \in \tilde{F}(\tilde{v}) := L_{i(\tilde{v})}$, this inequality yields

$$K_v \subset \bigcap_{i \in \mathcal{J}(v,v')} Q_{\bar{\gamma} r_i}(\tilde{f}(\tilde{v})) \subset \bigcap_{i \in \mathcal{J}(v,v')} (L_i + Q_{\bar{\gamma} r_i}).$$

Suppose now that $x^1(v, v') = 0$ (changing the coordinate system in $\mathbb{R}^n$, if any). Since $x^1(v, v')$ lies in $L_i \subset F(v)$, $i \in \mathcal{J}(v, v')$, we have

$$K_v \cap F(v) \subset H_{\bar{\gamma}}([\bigcap_{i \in \mathcal{J}(v,v')} (L_i + Q_{r_i})] \cap F(v))$$

where $H_{\bar{\gamma}}$ stands for dilation with respect to 0 by a factor of $\bar{\gamma}$. Applying (2.5) we then have

$$K_v \cap F(v) \subset H_{\bar{\gamma}}[(F(v') + x(v, v') + Q_{2\rho_\Gamma(v,v')}) \cap F(v)].$$

Since the vector $x^1(v, v')(= 0)$ belongs also to the shifted affine subspace $F(v') + x(v, v')$, the right-hand side equals $(F(v') + x(v, v') + Q_{2\bar{\gamma}\rho_\Gamma(v,v')}) \cap F(v)$ and the lemma follows. □

Given a cube $Q = Q_r(x)$ and $\lambda > 0$ we let λQ denote the cube $Q_{\lambda r}(x)$.

Lemma 2.8. $\hat{f}(v) \in \bar{\gamma} K_v$.

Proof. By (2.13) and (2.14)

$$\begin{aligned}\|\hat{f}(v) - f(v)\| &\le \|\hat{f}(v) - \tilde{f}(\tilde{v})\| + \|f(v) - \tilde{f}(\tilde{v})\| \\ &\le (1 + \sqrt{n})\gamma r_{i(\tilde{v})} = \bar{\gamma} r_{i(\tilde{v})}.\end{aligned}$$

Since K_v is centered at $f(v)$ and its radius (as ℓ^n_∞-ball) equals

$$\inf\{r_{i(\tilde{v})} : \tilde{v} \in \tilde{V}_\Gamma,\ \mathrm{pr}_1(\tilde{v}) = v\},$$

this inequality implies the statement of the lemma. □

Finally, we use Lemma 2.7 to find a point $y(v, v') \in F(v') + x(v, v')$ satisfying

$$\|f(v) - y(v, v')\| \le 2\bar{\gamma}\rho_\Gamma(v, v'). \tag{2.15}$$

Using this point we introduce an affine subspace $F(v, v')$ by shifting $F(v')$:

$$F(v, v') := F(v') + x(v, v') - y(v, v') + f(v). \tag{2.16}$$

Now we are in a position to prove (2.12) (recall that $v \leftrightarrow v'$ there). Using definition (2.9) we write

$$\|f(v) - f(v')\| \leq I_1 + I_2 + I_3$$

where we put

$$I_1 := \|\Pr(\hat{f}(v), F(v)) - \Pr(\hat{f}(v), F(v, v'))\|,$$
$$I_2 := \|\Pr(\hat{f}(v), F(v, v')) - \Pr(\hat{f}(v'), F(v, v'))\|,$$
$$I_3 := \|\Pr(\hat{f}(v'), F(v, v')) - \Pr(\hat{f}(v'), F(v'))\|.$$

Then we prove the desired estimates for I_k starting with I_2. Since orthogonal projections are non-expanding operators, we have by (2.11) the required inequality

$$I_2 \leq \|\hat{f}(v) - \hat{f}(v')\|_2 \leq \sqrt{n}\|\hat{f}(v) - \hat{f}(v')\| \leq \sqrt{n}\gamma\rho_\Gamma(v, v').$$

Here and below $\|\cdot\|_2$ stands for the Euclidean norm in $\mathbb{R}^n$.

To estimate I_3 note that by (2.16) $F(v, v')$ is a shift of $F(v')$ by the corresponding vector. Therefore,

$$I_3 \leq \|x(v, v') - y(v, v') + f(v)\|_2 \leq \sqrt{n}(\|x(v, v')\| + \|f(v) - y(v, v')\|).$$

Applying now (2.4) and (2.15) we obtain the required estimate

$$I_3 \leq \sqrt{n}(1 + 2\bar{\gamma})\rho_\Gamma(v, v').$$

For the remaining case we first note that (2.15), (2.16) and Lemma 2.7 yield

$$K_v \cap F(v) \subset F(v, v') + Q_{4\bar{\gamma}\rho_\Gamma(v,v')}.$$

Both sides of this embedding are central symmetric with respect to $f(v)$. Thus dilation with respect to $f(x)$ by a factor of $\lambda := \sqrt{n}\bar{\gamma}$ gives

$$(\lambda K_v) \cap F(v) \subset F(v, v') + Q_{4\lambda\bar{\gamma}\rho_\Gamma(v,v')}. \tag{2.17}$$

We let B denote the biggest Euclidean ball centered at $f(v)$ which is contained in λK_v. Clearly,

$$B \supset \bar{\gamma}K_v\ . \tag{2.18}$$

B_r also denotes the Euclidean ball with center 0 and radius $r := 4\sqrt{n}\lambda\bar{\gamma}\rho_\Gamma(v, v')$. Then (2.17) yields

$$B \cap F(v) \subset F(v, v') + B_r\ .$$

Note that the orthogonal projection of a point of $B \cap F(v)$ on $F(v, v')$ lies in $B \cap F(v, v')$. Together with the above embedding this observation leads to

$$B \cap F(v) \subset B \cap F(v, v') + B_r\ .$$

Without loss of generality we may assume that $\dim F(v) \geq \dim F(v') = \dim F(v, v')$. Since both of these affine subspaces pass through center $f(v)$ of the ball B, the embedding

$$B \cap F(v, v') \subset B \cap F(v) + B_r$$

holds as well.

The last two embeddings imply the following estimate of the Hausdorff distance:

$$d_H(B \cap F(v), B \cap F(v, v')) \le r := 4n(\bar{\gamma})^2 \rho_\Gamma(v, v'). \tag{2.19}$$

Recall that $d_H(A, A')$ for A, A' in $\mathbb{R}^n$ is defined by

$$d_H(A, A') := \inf\{s \ge 0 : \ A \subset A' + B_s, \ A' \subset A + B_s\}.$$

Here and below $B_s(x)$ is an Euclidean ball with center x and radius s and $B_s := B_s(0)$.

Now the required estimate of I_1 will follow from the next simple result.

Lemma 2.9. *Let L_1, L_2 be affine subspaces of $\mathbb{R}^n$ and let x be a point in $\mathbb{R}^n$ satisfying $y := \Pr(x, L_1) \in L_2$. Suppose that a ball $B_r(y)$ contains x. Then*

$$\|\Pr(x, L_1) - \Pr(x, L_2)\|_2 \le d_H(B_r(y) \cap L_1, B_r(y) \cap L_2). \tag{2.20}$$

Before proving the lemma we first finish the proof of Theorem 2.3. Set $L_1 := F(v)$ and $L_2 := F(v, v')$ and let $x := \hat{f}(v)$. Then by (2.9) $y := f(v) = \Pr(x, L_1)$ and by definition (2.16) $y \in L_2 = F(v, v')$. Now choose $B_r(y)$ to coincide with the ball B in (2.19). Then by (2.18) and Lemma 2.8, B contains $x := \hat{f}(v)$, so that the statement (2.20) can be applied to our settings. Thus we have

$$I_1 := \|\Pr(\hat{f}(v), F(v)) - \Pr(\hat{f}(v), F(v, v'))\| \le d_H(B \cap F(v), B \cap F(v, v')).$$

It remains to make use of (2.19) to prove

$$I_1 \le \tilde{\gamma}(k, n)\rho_\Gamma(v, v').$$

Proof of Lemma 2.9. Recall that $\|\cdot\|_2$ stands for the Euclidean norm in $\mathbb{R}^n$. Set $z := \Pr(x, L_2), \ w := \Pr(z, L_1)$. Then letting $\bar{r} := \|y - z\|_2$ we have

$$\|z - w\|_2 = \operatorname{dist}(z, B_{\bar{r}}(y) \cap L_1) \le d_H(B_{\bar{r}}(y) \cap L_1, B_{\bar{r}}(y) \cap L_2)$$

which implies

$$\|z - w\|_2 \le \frac{\|y - z\|_2}{r} d_H(B_r(y) \cap L_1, B_r(y) \cap L_2).$$

Let now $\mathcal{L}(A)$ denote the affine hull of a set $A \subset \mathbb{R}^n$. Put $L_3 := \mathcal{L}(\{x, y, z\})$ and denote by ℓ the straight line in L_3 going through y and orthogonal to $x - y$. Let $z' := \Pr(z, \ell)$. Then $z - z'$ is orthogonal to $\mathcal{L}(L_1 \cup \ell)$ so that

$$\|z - z'\|_2 \le \operatorname{dist}(z, L_1) = \|z - w\|_2.$$

Now from similarity of the rectangular triangles $\{y, z, z'\}$ and $\{x, y, z\}$ we have

$$\frac{\|y - z\|_2}{\|x - y\|_2} = \frac{\|z - z'\|_2}{\|y - z\|_2} \le \frac{\|z - w\|_2}{\|y - z\|_2} \le \frac{1}{r} d_H(B_r(y) \cap L_1, B_r(y) \cap L_2).$$

But $\|x - y\|_2 \le r$, since $x \in B_r(y)$, and the lemma follows. □

This finishes the proof of the direct part of Theorem 2.3.

The number 2^{k+1} from Theorem 2.3 in general cannot be decreased. This will follow from the proof of Theorem 1.3 given in Sections 3 and 4. Otherwise, the statement of Theorem 1.3 would hold for the finiteness number $N[C^{1,\omega}(\mathbb{R}^n)]$ strictly less than $3 \cdot 2^{n-1}$ which would contradict the result of Proposition 4.1.

Theorem 2.3 is completely proved. □

3. Proof of Theorem 1.3, part I

In this section we shall prove the inequality

$$N[C^{1,\omega}(\mathbb{R}^n)] \le 3 \cdot 2^{n-1}. \tag{3.1}$$

For this goal we associate with a set $X \subset \mathbb{R}^n$ a metric space

$$\mathcal{M}(X) := \{(x,y):\ x,y \in X,\ x \neq y\}$$

with a metric ρ_ω defined

$$\rho_\omega(m,m') := \omega(|m|) + \omega(|m'|) + \omega(\|m_x - m'_x\|) \tag{3.2}$$

for $m \neq m'$ and $\rho_\omega(m,m) := 0$. Here and below we use the following notation for $m = (x,y) \in \mathcal{M}(X)$:

$$|m| := \|x-y\|,\ m_x := x,\ m_y := y$$

where, recall, $\|x\| := \max\limits_{1\le i\le n} |x_i|$.

Note that since ω is concave, ρ_ω satisfies the triangle inequality. Then we can define the Lipschitz space $\mathrm{Lip}(\mathcal{M}(X), \mathbb{R}^n)$ by the seminorm (1.2).

Given $f : X \to \mathbb{R}^n$ define now a set-valued mapping $L_f : \mathcal{M}(X) \to \mathcal{A}_{n-1}(\mathbb{R}^n)$ by

$$L_f(m) := \{z \in \mathbb{R}^n : \langle z, m_x - m_y\rangle = f(m_x) - f(m_y)\} \tag{3.3}$$

where $\langle x,y\rangle := \sum\limits_{i=1}^n x_i y_i$.

Proposition 3.1. *A bounded function $f : X \to \mathbb{R}$ belongs to the trace space $C^{1,\omega}(\mathbb{R}^n)|_X$ if and only if there is a bounded Lipschitz selection $h : \mathcal{M}(X) \to \mathbb{R}^n$ of L_f. Moreover,*

$$\|f\|_{C^{1,\omega}(\mathbb{R}^n)|_X} \approx \|f\| + \inf_h\{\|h\| + |h|_{\mathrm{Lip}}\} \tag{3.4}$$

where[2] $\|f\| := \sup\limits_{x\in X} |f(x)|$, $\|h\| := \sup\limits_{m\in\mathcal{M}(X)} \|h(m)\|$ *and* $\mathrm{Lip} := \mathrm{Lip}(\mathcal{M}(X), \mathbb{R}^n)$.

Proof. Note that we obtain an equivalent norm substituting in (1.1) ω for $\min(1,\omega)$. So we can assume that

$$\omega \le 1. \tag{3.5}$$

Therefore, for our goal the following equivalent norm

$$\|f\|_{C^{1,\omega}(\mathbb{R}^n)} := \|f\| + \|\nabla f\| + \sup_{x\neq y} \frac{\|\nabla f(x) - \nabla f(y)\|}{\omega(\|x-y\|)}$$

will be more appropriate.

(*Necessity*). Let $f \in C^{1,\omega}(\mathbb{R}^n)|_X$. Without loss of generality we may assume that $\|f\|_{C^{1,\omega}(\mathbb{R}^n)|_X} < 1$. Then there is a function $F \in C^{1,\omega}(\mathbb{R}^n)$ with $F|_X = f$ such that

$$\|F\|_{C^{1,\omega}(\mathbb{R}^n)} < 1. \tag{3.6}$$

Set $g(x) := \nabla F(x),\ x \in X$ and define the required mapping $h : \mathcal{M}(X) \to \mathbb{R}^n$ by the condition $h(m) \in L_f(m)$ and

$$\|h(m) - g(m_x)\|_2 = \mathrm{dist}_{\ell_2^n}(g(m_x), L_f(m)) \tag{3.7}$$

[2] $f \approx g$ means that $\gamma_1 f \le g \le \gamma_2 f$ for some positive constants γ_1, γ_2 depending only on n.

for $m \in \mathcal{M}(X)$. Then h is a selection of L_f and by definitions (3.7) and (3.3)

$$\begin{aligned}\|h(m) - g(m_x)\| &\le \|h(m) - g(m_x)\|_2 \\ &= |f(m_x) - f(m_y) - \langle g(m_x), m_x - m_y\rangle|/\|m_x - m_y\|_2 \\ &= |F(m_x) - F(m_y) - \langle \nabla F(m_x), m_x - m_y\rangle|/\|m_x - m_y\|_2.\end{aligned}$$

By the Taylor formula, (3.5) and (3.6) the right-hand side does not exceed

$$\left(\sup_{x\neq y} \frac{\|\nabla F(x) - \nabla F(y)\|_2}{\omega(\|x-y\|)}\right)\omega(|m|) \le \sqrt{n}\|F\|_{C^{1,\omega}(\mathbb{R}^n)}\omega(|m|) \le \sqrt{n}.$$

Hence we have

$$\|h(m)\| \le \|g(m_x)\| + \|h(m) - g(m_x)\| \le \|\nabla F(m_x)\| + \sqrt{n} \le 1 + \sqrt{n},$$

so that $\|h\| \le 1 + \sqrt{n}$.

It remains to estimate $|h|_{\mathrm{Lip}}$. Let $m \neq m'$. Then

$$\|h(m) - h(m')\| \le \|h(m) - g(m_x)\| + \|h(m') - g(m'_x)\| + \|g(m_x) - g(m'_x)\|.$$

Estimating each term on the right as above and applying definition (3.2) we obtain

$$\|h(m) - h(m')\| \le \sqrt{n}(\omega(|m|) + \omega(|m'|)) + \omega(\|m_x - m'_x\|) \le \sqrt{n}\rho_\omega(m, m')$$

so that $|h|_{\mathrm{Lip}} \le \sqrt{n}$.

(*Sufficiency*). Let a function $f : X \to \mathbb{R}$ and a mapping $h : \mathcal{M}(X) \to \mathbb{R}^n$ satisfy the condition of the proposition. Without loss of generality we may assume that

$$\|f\| + \|h\| + |h|_{\mathrm{Lip}} < 1. \tag{3.8}$$

We have to prove that f then belongs to $C^{1,\omega}(\mathbb{R}^n)|_X$ and has a trace norm bounded by a constant depending only on n. For this goal we apply the Whitney extension theorem (see, e.g., [St], Ch. 6) which in our settings states that f has the required properties if there is a 1-jet $g = (g_1, ..., g_n) : X \to \mathbb{R}^n$ so that for some constant $\gamma = \gamma(n)$

$$\|g\| + \sup_{x,y\in X,\ x\neq y} \frac{\|g(x) - g(y)\|}{\omega(\|x-y\|)} \le \gamma(n); \tag{3.9}$$

$$\sup_{x,y\in X,\ x\neq y} \frac{|f(x) - f(y) - \langle g(x), x-y\rangle|}{\|x-y\|\omega(\|x-y\|)} \le \gamma(n). \tag{3.10}$$

So it suffices to find $g : X \to \mathbb{R}^n$ satisfying these conditions. To this end for an isolated point $x \in X$ we let $\hat{x}$ denote a nearest to x point in $X\backslash\{x\}$ (measuring distance in the norm $\|\cdot\|$); otherwise, we put $\hat{x} := x$. Then we set

$$g(x) := \begin{cases} h((x,\hat{x})), & \text{if } x \text{ is an isolated point of } X, \\ \lim\limits_{y\to x, y\in X} h((x,y)), & \text{otherwise.}\end{cases} \tag{3.11}$$

Existence of the limit follows from (3.8), since

$$\|h((x,y')) - h((x,y''))\| \le \rho_\omega((x,y'),(x,y'')) = \omega(\|x-y'\|) + \omega(\|x-y''\|).$$

By the definitions (3.11) and (3.8) we also have $\|g\| \le \|h\| < 1$.

Choose for every $x \in X$ a sequence (maybe, stationary) $\{x_i\} \subset X\backslash\{x\}$ so that

$$\hat{x} = \lim_{i\to\infty} x_i \text{ and } g(x) = \lim_{i\to\infty} h((x,x_i)). \tag{3.12}$$

Such a sequence exists by definition of g. Set $m_i(x) := (x, x_i)$ $(\in \mathcal{M}(X))$.

Now by (3.8),(3.12) and the definition of $\hat{x}$ we have

$$\begin{aligned}\|g(x)-g(x')\| &= \lim_{i\to\infty}\|h(m_i(x))-h(m_i(x'))\|\\ &\le |h|_{\mathrm{Lip}}\lim_{i\to\infty}\rho_\omega(m_i(x),m_i(x'))\\ &\le \omega(\|x-\hat{x}\|)+\omega(\|x'-\hat{x}'\|)+\omega(\|x-x'\|)\le 3\omega(\|x-x'\|).\end{aligned}$$

So we have proved (3.9) with the constant 4. It remains to check (3.10). Since $h(m)\in L_f(m)$, by (3.3) the left-hand side of (3.10) equals

$$\frac{|\langle h(m)-g(x),x-y\rangle|}{\omega(\|x-y\|)\|x-y\|}=\lim_{i\to\infty}\frac{|\langle h(m)-h(m_i(x)),x-y\rangle|}{\omega(|m|)|m|}$$

where we set $m:=(x,y)$.

By the Cauchy inequality the numerator on the right

$$\le n|m|\limsup_{i\to\infty}\|h(m)-h(m_i(x))\|$$

which, in turn, does not exceed

$$\begin{aligned}n|m||h|_{\mathrm{Lip}}\limsup_{i\to\infty}\rho_\omega(m,m_i(x)) &= n|m||h|_{\mathrm{Lip}}(\omega(|m|)+\omega(\|x-\hat{x}\|))\\ &\le 2n|h|_{\mathrm{Lip}}|m|\omega(|m|).\end{aligned}$$

Finally, applying (3.8) we estimate the left-hand side of (3.10) by $2n$. □

Now we widen the metric space $\mathcal{M}(X)$ by a point $*$. Set $\tilde{\mathcal{M}}(X):=\mathcal{M}(X)\bigcup\{*\}$ and extend the metric ρ_ω to $\tilde{\mathcal{M}}(X)$ by

$$\tilde{\rho}_\omega(m,m'):=\begin{cases}\rho_\omega(m,m'), & m,m'\in\mathcal{M}(X),\\ 2, & \text{otherwise.}\end{cases}$$

We also extend the set-valued mapping $L_f:\mathcal{M}(X)\to\mathcal{A}_{n-1}(\mathbb{R}^n)$ by

$$\tilde{L}_f(m):=\begin{cases}L_f(m), & m\in\mathcal{M}(X),\\ \{0\}, & m=*\ .\end{cases}$$

Thus $\tilde{L}_f$ maps $\tilde{\mathcal{M}}(X)$ into the set $\mathcal{A}(\mathbb{R}^n)$ of all affine subspaces of $\mathbb{R}^n$.

In this setting the statement of Proposition 3.1 can be reformulated in the following way.

Proposition 3.2. *A bounded function $f:X\to\mathbb{R}$ belongs to $C^{1,\omega}(\mathbb{R}^n)|_X$ iff $\tilde{L}_f$ has a Lipschitz (with respect to $\tilde{\rho}_\omega$) selection $\tilde{h}:\tilde{\mathcal{M}}(X)\to\mathbb{R}^n$. Moreover,*

$$\|f\|_{C^{1,\omega}(\mathbb{R}^n)|_X}\approx\|f\|+\inf_{\tilde{h}}|\tilde{h}|_{\mathrm{Lip}(\tilde{\mathcal{M}}(X),\mathbb{R}^n)}.$$

Now equip $\tilde{\mathcal{M}}(X)$ with a weighted graph structure as follows. The set of vertices of this graph $\Gamma(X)$ coincides with $\tilde{\mathcal{M}}(X)$. We join two different points $m,m'\in\mathcal{M}(X)$ by an edge, if $\{m_x,m_y\}\bigcap\{m'_x,m'_y\}\ne\emptyset$. We also let $*$ be joined with every $m\in\mathcal{M}(X)$. If m and m' are joined by an edge ($m\leftrightarrow m'$, in short), then we define a weight by

$$w_{\Gamma(X)}(m,m'):=\begin{cases}\omega(|m|)+\omega(|m'|), & \text{if } m,m'\in\mathcal{M}(X),\\ 2, & \text{otherwise.}\end{cases}\tag{3.13}$$

At last we define a (pseudo-)metric $\rho_{\Gamma(X)}$ by (2.1), i.e.,

$$\rho_{\Gamma(X)}(m, m') := \inf \sum_{i=0}^{k-1} w_{\Gamma(X)}(m_i, m_{i+1}) \tag{3.14}$$

where the infimum is taken over all finite paths $\{m_i\}_{i=0}^{k}$ joining m and m' (i.e., $m_0 = m$, $m_k = m'$ and $m_i \leftrightarrow m_{i+1}$).

In fact, $\rho_{\Gamma(X)}$ is a metric equivalent to $\tilde{\rho}_\omega$ as in the following:

Proposition 3.3. $\frac{1}{2}\tilde{\rho}_\omega \leq \rho_{\Gamma(X)} \leq 2\tilde{\rho}_\omega$.

Proof. In the trivial case $m \in \mathcal{M}(X)$ and $m' = *$ we have by the definitions

$$\rho_\omega(m, m') = 2 = \rho_{\Gamma(X)}(m, m').$$

Otherwise, we set $m'' := (m_x, m'_x)$. Then $\{m, m'', m'\}$ is a path connecting m and m' in $\mathcal{M}(X)$ and by (3.13) and (3.14) we have

$$\begin{aligned}\rho_{\Gamma(X)}(m, m') &\leq w_{\Gamma(X)}(m, m'') + w_{\Gamma(X)}(m'', m') \\ &\leq 2(\omega(|m|) + \omega(|m'|) + \omega(\|m_x - m'_x\|)) = 2\tilde{\rho}_\omega(m, m').\end{aligned}$$

To prove the inverse inequality note first, that if a path $\{m_i\}_{i=0}^{k}$ joining m and m' satisfies $m_i = *$ for some $0 < i < k$, then

$$\sum_{i=0}^{k-1} w_{\Gamma(X)}(m_i, m_{i+1}) \geq 4 > \omega(|m|) + \omega(|m'|) + \omega(\|m_x - m'_x\|) = \tilde{\rho}_\omega(m, m').$$

(recall that $\omega \leq 1$; see (3.5)).

On the other hand, if $\{m_i\}_{i=0}^{k} \subset \mathcal{M}(X)$, then by subadditivity of ω we have

$$\begin{aligned}\sum_{i=0}^{k-1} w_{\Gamma(X)}(m_i, m_{i+1}) &= \omega(|m|) + 2\sum_{i=1}^{k-1} \omega(|m_i|) + \omega(|m'|) \\ &\geq \frac{1}{2}(\omega(|m|) + \omega(|m'|) + \omega(\sum_{i=0}^{k} |m_i|)).\end{aligned}$$

But $m_i \leftrightarrow m_{i+1}$ and therefore

$$\sum_{i=0}^{k} |m_i| \geq \|m_x - m'_x\|.$$

Hence the right-hand side of the previous inequality is

$$\geq \frac{1}{2}(\omega(|m|) + \omega(|m'|) + \omega(\|m_x - m'_x\|)) = \frac{1}{2}\tilde{\rho}_\omega(m, m')$$

and the proposition follows. □

Corollary 3.4. *Proposition 3.2 holds with* $(\tilde{\mathcal{M}}(X), \rho_{\Gamma(X)})$ *instead of* $(\tilde{\mathcal{M}}(X), \tilde{\rho}_\omega)$.

To finish the proof of the theorem we need one more auxiliary result.

Proposition 3.5. *Let f be a function defined on X and $\mathcal{N}$ be an admissible subset of $\tilde{\mathcal{M}}(X)$ (see Definition 2.1) with* $\text{card}(\mathcal{N}) \leq \frac{2}{3}m$ *for some fixed integer m. If the restriction $f|_Y$ to every subset $Y \subset X$ consisting of at most m points has an extension f_Y satisfying*

$$\|f_Y\|_{C^{1,\omega}(\mathbb{R}^n)} \leq 1, \tag{3.15}$$

then the set-valued mapping $\tilde{L}_f|_{\mathcal{N}} : \mathcal{N} \to \mathcal{A}(\mathbb{R}^n)$ *has a Lipschitz selection* $h_{\mathcal{N}}$ *such that*

$$|h_{\mathcal{N}}|_{\mathrm{Lip}(\mathcal{N},\mathbb{R}^n)} \le \gamma(n).$$

Proof. First note that if Γ is a graph with p vertices and r edges having no isolated edges, then

$$p \le \frac{3}{2}r. \tag{3.16}$$

Consider now the set

$$X_{\mathcal{N}} := \{z \in X : \ \exists\, m \in \mathcal{N} \text{ so that } m_x = z \text{ or } m_y = z\}$$

and check that

$$\mathrm{card}(X_{\mathcal{N}}) \le m. \tag{3.17}$$

Equip $X_{\mathcal{N}}$ with the graph structure induced by $\mathcal{N}$, i.e., $\{x, y\} \subset X_{\mathcal{N}}$ defines an edge if $(x, y) \in \mathcal{N}$. Since $\mathcal{N}$ is admissible, $X_{\mathcal{N}}$ has no isolated edges. Therefore, for $\Gamma = X_{\mathcal{N}}$ the number of vertices is $p = \mathrm{card}(X_{\mathcal{N}})$ and the number of edges is $r \le \mathrm{card}(\mathcal{N}) \le \frac{2}{3}m$, so that inequality (3.17) follows from (3.16).

Applying now assumption (3.15) of the proposition to $Y := X_{\mathcal{N}}$ one can state that the function $g := f|_{X_{\mathcal{N}}}$ satisfies

$$\|g\|_{C^{1,\omega}(\mathbb{R}^n)|_{X_{\mathcal{N}}}} \le 1.$$

Then from the necessary conditions of Corollary 3.4 applied to $X_{\mathcal{N}}$ and g, one derives that the set-valued mapping $\tilde{L}_g : \mathcal{M}(X_{\mathcal{N}}) \bigcup \{*\} \to \mathcal{A}(\mathbb{R}^n)$ has a Lipschitz selection $h : \mathcal{M}(X_{\mathcal{N}}) \bigcup \{*\} \to \mathbb{R}^n$ with Lipschitz seminorm $\le \gamma(n)$.

On the other hand, $\mathcal{N} \subset \mathcal{M}(X_{\mathcal{N}}) \bigcup \{*\}$ and therefore $\tilde{L}_f|_{\mathcal{N}} = \tilde{L}_g|_{\mathcal{N}}$. If we now set $h_{\mathcal{N}} := h|_{\mathcal{N}}$, then $h_{\mathcal{N}}$ will be the required Lipschitz selection of $\tilde{L}_f|_{\mathcal{N}}$. □

We are now in a position to prove inequality (3.1). Let $f : X \to \mathbb{R}$ be a function satisfying the finiteness condition: the restriction $f|_Y$ to an arbitrary subset $Y \subset X$ of cardinality $\le m := 3 \cdot 2^{n-1}$ has an extension f_Y with

$$\|f_Y\|_{C^{1,\omega}(\mathbb{R}^n)} \le 1. \tag{3.18}$$

We have to prove that $f \in C^{1,\omega}(\mathbb{R}^n)|_X$ and

$$\|f\|_{C^{1,\omega}(\mathbb{R}^n)|_X} \le \gamma(n). \tag{3.19}$$

For this goal it suffices to check that f satisfies the (sufficient) assumptions of Corollary 3.4. Applying first (3.18) to subsets $Y \subset X$ with $\mathrm{card}\, Y = 1$ we have $\sup_X |f| \le 1$. Then by (3.18) and Proposition 3.5 for every admissible subset $\mathcal{N} \subset \tilde{\mathcal{M}}(X)$ of cardinality

$$\mathrm{card}(\mathcal{N}) \le \frac{2}{3}m = 2^n$$

the set-valued mapping $\tilde{L}_f|_{\mathcal{N}}$ has a Lipschitz selection $h_{\mathcal{N}}$ with

$$|h_{\mathcal{N}}|_{\mathrm{Lip}(\mathcal{N},\mathbb{R}^n)} \le \gamma(n).$$

Thus the assumptions of Theorem 2.3 are fulfilled for $\tilde{L}_f$. Hence $\tilde{L}_f$ has a Lipschitz selection $h : \tilde{\mathcal{M}}(X) \to \mathbb{R}^n$ with

$$|h|_{\mathrm{Lip}(\tilde{\mathcal{M}}(X),\mathbb{R}^n)} \le \gamma_1(n).$$

Thus f satisfies the assumptions of Corollary 3.4 so that (3.19) holds.

The proof of inequality (3.1) is complete.

Remark 3.6. The result holds for the homogeneous space $\dot{C}^{1,\omega}(\mathbb{R}^n)$ as well. Recall that it is defined by the seminorm $\sup_{x\neq y}\|\nabla f(x)-\nabla f(y)\|/\omega(\|x-y\|)$. The small changes in the proof leading to this result may be left to the reader.

4. Proof of Theorem 1.3, part II

In the previous section we have proved that $N[C^{1,\omega}(\mathbb{R}^n)] \le 3\cdot 2^{n-1}$. It remains to prove the inverse inequality which, clearly, follows from the next result.

Proposition 4.1. *There exist a compact $X = X_\omega$ in $\mathbb{R}^n$ and a function $F = F_\omega : X \to \mathbb{R}$ such that the restriction $F|_Y$ to every subset $Y \subset X$ with* card$(Y) \le 3\cdot 2^{n-1}-1$ *has an extension to a function F_Y satisfying $\|F_Y\|_{C^{1,\omega}(\mathbb{R}^n)} \le 1$, while $F \notin C^{1,\omega}(\mathbb{R}^n)|_X$.*

Proof. We define the required set $X = X_\omega$ as a union of pairwise disjoint sets $X^{(m)},\ m = 1,2,...,$

$$X = \bigcup_{m=1}^{\infty} X^{(m)}.$$

The corresponding function $F = F_\omega : X \to \mathbb{R}$ is defined by

$$F|_{X^{(m)}} := F^{(m)},\ m = 1,2,....$$

The definitions and properties of the sets $X^{(m)}$ and the functions $F^{(m)}$ will be presented in the following chain of results.

We begin with the following inductive procedure to determine subsets

$$X_k \subset L_k := \{x \in \mathbb{R}^n :\ x_i = 0,\ k+1 \le i \le n\},$$

points $\{a^{(k)}, b^{(k)}\} \subset X_k$, and functions $f_k : X_k \to \mathbb{R}$ where $k = 1,2,\dots,n$.

Let $\{t_k\}_{k=1}^n,\ \{r_k\}_{k=0}^n$ be positive monotone sequences *decreasing* and *increasing* respectively, satisfying

$$t_1 = r_0 \text{ and } 8r_k \le r_{k+1} \le 1\ (k = 0,1,\dots,n-1). \tag{4.1}$$

Put $X_1 := \{0, t_1\vec{e}_1, 2t_1\vec{e}_1\}$ and $a^{(1)} = b^{(1)} := t_1\vec{e}_1$. Here $\{\vec{e}_1,\dots,\vec{e}_n\}$ stands for the canonical basis in $\mathbb{R}^n$.

If the set $X_k \subset L_k$ and the points $a^{(k)}, b^{(k)}$ have been already constructed, then we set

$$b^{(k+1)} := a^{(k)} + t_{k+1}\vec{e}_{k+1},\ a^{(k+1)} := 2r_k\vec{e}_1 - b^{(k+1)}, \tag{4.2}$$

$$X_{k+1} := \tilde{X}_{k+1} \cup \{2r_k\vec{e}_1 - \tilde{X}_{k+1}\} \tag{4.3}$$

where

$$\tilde{X}_{k+1} := (X_k \backslash \{a^{(k)}\}) \cup \{b^{(k+1)}\}. \tag{4.4}$$

Clearly, $X_{k+1} \subset L_{k+1}$ and $\{a^{(k+1)}, b^{(k+1)}\} \subset X_{k+1}$. Moreover, for every $x \in X_k$ we have $0 \le x_1 \le 2r_{k-1}$. Hence by (4.1) and (4.4) it follows that for each $x \in\ 2r_k\vec{e}_1 - \tilde{X}_{k+1}$

$$x_1 \ge r_k. \tag{4.5}$$

Finally, we put

$$f_k(x) := \begin{cases} \min\limits_{i=1,\dots,n} t_i\omega(r_i), & \text{if } x = b^{(k)}, \\ 0, & \text{otherwise.} \end{cases} \tag{4.6}$$

Lemma 4.2. *Let $y \in X_k$. There is a function $f_{k,y} \in C^{1,\omega}(\mathbb{R}^n)$ satisfying*

(i) *$f_{k,y}(x) = f_k(x)$ for all $x \in X_k \backslash \{y\}$;*

(ii) *$\operatorname{supp} f_{k,y} \subset \{x \in \mathbb{R}^n : |x_1| \le \frac{1}{2}r_k\}$;*

(iii) *$\|f_{k,y}\|_{C^{1,\omega}(\mathbb{R}^n)} \le \gamma(n)$.*

Proof. Recall that given $x \in \mathbb{R}^n$ and $r > 0$ we let $Q_r(x)$ denote the cube $\{y \in \mathbb{R}^n : \|y - x\| \le r\}$.

Lemma 4.3. *For every $0 < r \le 1$, $k \in \{1, \dots, n\}$ and $a \in \mathbb{R}^n$, such that $\|a\| \le r$ and $a_k \ne 0$, there is a function $\varphi = \varphi_{k,r,a} : \mathbb{R}^n \to \mathbb{R}$ such that*

(a) $\varphi(a) = 1$;

(b) $\varphi(-x) = -\varphi(x)$ if $x \in Q_r(0)$;

(c) $\varphi(x) = 0$ if $x \in L_{k-1} \cap Q_r(0)$ or $x \notin Q_{2r}(0)$;

(d) $\varphi \in C^{1,\omega}(\mathbb{R}^n)$ and $\|\varphi\|_{C^{1,\omega}(\mathbb{R}^n)} \le \gamma/(|a_k|\omega(r))$, where γ depends on n only.

Proof. Put

$$\psi(t) := \begin{cases} t(2r - |t|)^2/(a_k(2r - |a_k|)^2), & \text{if } |t| \le 2r, \\ 0, & \text{otherwise.} \end{cases}$$

Then introduce

$$\tilde{\varphi}(x) := \begin{cases} \psi(x_k), & \text{if } \|x\| \le r, \\ 0, & \text{if } \|x\| \ge 2r. \end{cases}$$

It can be readily seen that the function $\tilde{\varphi}$ defined on $\{x \in \mathbb{R}^n : \|x\| \le r \text{ or } \|x\| \ge 2r\}$ satisfies on this set the conditions of the Whitney extension theorem (see, e.g., [St], Ch. 6) and its trace norm does not exceed $\gamma_1(n)/(|a_k|\omega(r))$. Thus $\tilde{\varphi}$ can be extended to a function $\varphi : \mathbb{R}^n \to \mathbb{R}$ satisfying condition (d) of the lemma. Validity of conditions (a)–(c) follows from the definition of the function $\tilde{\varphi}$. □

We proceed the proof of Lemma 4.2 by induction on k. Check, first, that the lemma is true for $k = 1$.

In this case $X_1 = \{0, t_1\vec{e}_1, 2t_1\vec{e}_1\}$, $a^{(1)} = b^{(1)} = t_1\vec{e}_1$, $f_1(0) = f_1(2t_1\vec{e}_1) = 0$ and $f_1(t_1\vec{e}_1) = \min\limits_{i=1,\dots,n}\{t_i\omega(r_i)\}$. Therefore, the result is evident, if $y = t_1\vec{e}_1$. Consider the cases $y = 0$ and $y = 2t_1\vec{e}_1$. Define a function $f_{1,y} : \mathbb{R}^n \to \mathbb{R}$ by

$$f_{1,y}(x) := f_1(t_1\vec{e}_1)\varphi(x - y),$$

where $\varphi = \varphi_{k,r,a}$ is the function of Lemma 4.3 with $k := 1$, $r := \frac{1}{8}r_1$ and $a := t_1\vec{e}_1 - y$. Then conditions (a)-(c) of this lemma and inequality $8t_1 \le r_1$ (see (4.1)) imply conditions (i) and (ii) of Lemma 4.2. To prove (iii) we apply Lemma 4.3(d) and get

$$\|f_{1,y}\|_{C^{1,\omega}(\mathbb{R}^n)} = f_1(t_1\vec{e}_1)\|\varphi\|_{C^{1,\omega}(\mathbb{R}^n)} \le \gamma(n)\,\frac{\min\limits_{i=1,\dots,n} t_i\omega(r_i)}{t_1\omega(\frac{1}{8}r_1)} \le \gamma_1(n).$$

Suppose now that the statement is valid for $k > 1$ and show that the same is true for $k + 1$.

Lemma 4.4. *The statements of Lemma 4.2 hold for $y \in \tilde{X}_{k+1}$. Moreover, the function $f_{k+1,y}$ can be determined in such a way that $f_{k+1,y}(x) = 0$, if $|x_1| \ge r_k$.*

Proof. Put $f_{k+1,y} = 0$ if $y = b^{(k+1)} \in \tilde{X}_{k+1}$; see (4.4).

Now let $y \in \tilde{X}_{k+1}$ but $y \neq b^{(k+1)}$. Set

$$y^* := 2r_{k-1}\vec{e}_1 - y \ .$$

By the induction assumption there is a function $\tilde{f} = f_{k,y^*} \in C^{1,\omega}(\mathbb{R}^n)$ with

$$\|\tilde{f}\|_{C^{1,\omega}(\mathbb{R}^n)} \ \leq \ \gamma_1(n) \tag{4.7}$$

such that $\tilde{f}$ vanishes both on the set $X_k \backslash \{y^*, b^{(k)}\}$ and $\{x \in \mathbb{R}^n : \ |x_1| \geq \frac{1}{2}r_k\}$ and satisfies

$$\tilde{f}(b^{(k)}) \ = \ \min_{i=1,\dots,n} t_i\omega(r_i).$$

Define the promised function $f_{k+1,y}$ by setting

$$f_{k+1,y}(x) := \tilde{f}(2r_{k-1}\vec{e}_1 - \mathrm{Pr}_k(x)). \tag{4.8}$$

where $\mathrm{Pr}_k(x) := (x_1, \dots, x_k, 0, \dots, 0)$.

Check that $f_{k+1,y}$ satisfies conditions (i)-(iii) of Lemma 4.2. The latter one immediately follows from (4.7) and (4.8). Let us check (i). Since $a^{(k)} \in L_k$ (i.e., $a_i^{(k)} = 0, \ i \geq k+1$) and $b^{(k+1)} = a^{(k)} + t_{k+1}\vec{e}_{k+1}$, we have

$$\begin{aligned} f_{k+1,y}(b^{(k+1)}) &= \tilde{f}(2r_{k-1}\vec{e}_1 - \mathrm{Pr}_k(b^{(k+1)})) = \tilde{f}(2r_{k-1}\vec{e}_1 - a^{(k)}) \\ &= \tilde{f}(b^{(k)}) = \min_{i=1,\dots,n} t_i\omega(r_i). \end{aligned}$$

On the other hand, $\tilde{X}_{k+1} \backslash \{b^{(k+1)}, y\} = X_k \backslash \{a^{(k)}, y\}$ by (4.3) and (4.4) and

$$2r_{k-1}\vec{e}_1 - \{X_k \backslash \{a^{(k)}, y\}\} = X_k \backslash \{b^{(k)}, y^*\}.$$

Combining with (4.8) one gets

$$f_{k+1,y}|_{\tilde{X}_{k+1} \backslash \{b^{(k+1)}, y\}} = \tilde{f}|_{X_k \backslash \{b^{(k)}, y^*\}} = 0. \tag{4.9}$$

Finally, $\tilde{f}$ vanishes on the set $\{x \in \mathbb{R}^n : \ |x_1| \geq \frac{1}{2}r_k\}$ so that $f_{k+1,y}$ vanishes on the set $\{x \in \mathbb{R}^n : \ |x_1| \geq r_{k-1} + \frac{1}{2}r_k\}$. Since $r_{k-1} \leq \frac{1}{8}r_k$ (see (4.1)) we get, in particular, that

$$f_{k+1,y}|_{\{x \in \mathbb{R}^n : |x_1| \geq r_k\}} = 0. \tag{4.10}$$

From this and (4.5) it follows that $f_{k+1,y}$ vanishes on the set $2r_k\vec{e}_1 - \tilde{X}_{k+1}$. Together with (4.9) this implies (i).

Since $r_k \leq \frac{1}{2}r_{k+1}$, condition (4.10) also implies (ii), and the lemma follows. □

Lemma 4.5. *The statements of Lemma 4.2 hold for every* $y \in 2r_k\vec{e}_1 - \tilde{X}_{k+1}$.

Proof. Put $\tilde{y} := 2r_k\vec{e}_1 - y$. Then $\tilde{y} \in \tilde{X}_{k+1}$ and by Lemma 4.4 there is a function $\tilde{f} = f_{k+1,\tilde{y}} \in C^{1,\omega}(\mathbb{R}^n)$ satisfying

$$\|\tilde{f}\|_{C^{1,\omega}(\mathbb{R}^n)} \leq \gamma_1(n), \tag{4.11}$$

$$\tilde{f}(b_{k+1}) \ = \ \min_{i=1,\dots,n} t_i\omega(r_i) \tag{4.12}$$

and vanishing on the set $(X_{k+1} \backslash \{b_{k+1}, \tilde{y}\}) \cup \{x \in \mathbb{R}^n : |x_1| \geq r_k\}$.

Let $r = \frac{1}{8}r_{k+1},\ a = a^{(k+1)} - r_k\vec{e}_1$. Denote by φ the function $\varphi_{k+1,r,a}$ of Lemma 4.3 and put

$$\tilde{\varphi}(x) = \tilde{f}(b_{k+1})\,\varphi(x - r_k\vec{e}_1)\ (= \min_{i=1,\dots,n}\{t_i\omega(r_i)\}\,\varphi(x - r_k\vec{e}_1)). \tag{4.13}$$

Finally, define the required function $f_{k+1,y}$ by

$$f_{k+1,y}(x) := (\tilde{f} + \tilde{\varphi})(2r_k\vec{e}_1 - x). \tag{4.14}$$

Let us show that it satisfies the conditions of Lemma 4.2. Note that

$$2r_k\vec{e}_1 - b^{(k+1)} = a^{(k+1)},$$

see (4.2), and $\varphi(a) = 1$ by Lemma 4.3. Hence we have

$$\begin{aligned} f_{k+1,y}(b^{(k+1)}) &= \tilde{f}(a^{(k+1)}) + \tilde{\varphi}(a^{(k+1)}) = \tilde{\varphi}(a^{(k+1)}) \\ &= \min_{i=1,\dots,n}\{t_i\omega(r_i)\}\,\varphi(a^{(k+1)} - r_k\vec{e}_1) \\ &= \min_{i=1,\dots,n}\{t_i\omega(r_i)\}\,\varphi(a) = \min_{i=1,\dots,n} t_i\omega(r_i). \end{aligned}$$

To check condition (i) of Lemma 4.2 it remains to prove that $f_{k+1,y}$ equals 0 on $X_{k+1}\backslash\{b^{(k+1)}, y\}$. It follows from (4.2)-(4.4) that

$$X_{k+1}\backslash\{a^{(k+1)}, b^{(k+1)}\} \subset L_k \cap Q_{r_k}(r_k\vec{e}_1). \tag{4.15}$$

On the other hand, by Lemma 4.3(c) and the inequality $r_k \le \frac{1}{8}r_{k+1}$ the function $\tilde{\varphi}$ equals 0 on $L_k \cap Q_{\frac{1}{8}r_{k+1}}(r_k\vec{e}_1) \supset L_k \cap Q_{r_k}(r_k\vec{e}_1)$. Together with (4.15), this implies

$$\tilde{\varphi}(2r_k\vec{e}_1 - \cdot)|_{X_{k+1}\backslash\{a^{(k+1)},b^{(k+1)}\}} = \tilde{\varphi}|_{X_{k+1}\backslash\{a^{(k+1)},b^{(k+1)}\}} = 0.$$

Besides, $\tilde{f}$ vanishes on $X_{k+1}\backslash\{b^{(k+1)}, \tilde{y}\}$ and therefore $\tilde{f}(2r_k\vec{e}_1 - x) = 0$ for $x \in X_{k+1}\backslash\{a^{(k+1)}, y\}$. Along with (4.13) and (4.15) this leads to the equality $f_{k+1,y}(x) = 0$, if $x \in X_{k+1}\backslash\{a^{k+1}, b^{(k+1)}, y\}$. By (4.6) we then have

$$f_{k+1,y}(x) \ = \ f_{k+1}(x)\ (= 0),\ x \in X_{k+1}\backslash\{a^{(k+1)}, b^{(k+1)}, y\}.$$

It remains to check that the previous equality holds true for $x = a^{(k+1)}$ ($\ne y$). But according to statements (a) and (b) of Lemma 4.3 and by (4.12) we have

$$\begin{aligned} f_{k+1,y}(a^{(k+1)}) &= \tilde{f}(b^{(k+1)}) + \tilde{\varphi}(b^{(k+1)}) \\ &= \tilde{f}(b^{(k+1)}) + \tilde{f}(b^{(k+1)})\varphi(b^{(k+1)} - r_k\vec{e}_1) \\ &= \tilde{f}(b^{(k+1)}) - \tilde{f}(b^{(k+1)}) = 0. \end{aligned}$$

Thus condition (i) holds.

Now condition (ii) immediately follows from (4.13), (4.14), Lemma 4.3(c) and the equality $\tilde{f}(x) = 0$, if $|x_1| \ge r_k$.

For the proof of (iii) note that by (4.13) and (4.14)

$$\begin{aligned} \|f_{k+1,y}\|_{C^{1,\omega}(\mathbb{R}^n)} &\le \|\tilde{f}(2r_k\vec{e}_1 - \cdot)\|_{C^{1,\omega}(\mathbb{R}^n)} + \|\tilde{\varphi}(2r_k\vec{e}_1 - \cdot)\|_{C^{1,\omega}(\mathbb{R}^n)} \\ &\le \|\tilde{f}\|_{C^{1,\omega}(\mathbb{R}^n)} + \min_{i=1,\dots,n}\{t_i\omega(r_i)\}\|\varphi\|_{C^{1,\omega}(\mathbb{R}^n)}. \end{aligned}$$

Together with (4.11) and Lemma 4.3(d) this gives

$$\|f_{k+1,y}\|_{C^{1,\omega}(\mathbb{R}^n)} \le \gamma_1(n) + \gamma_2(n)\ \frac{\min\limits_{i=1,\ldots,n} t_i\omega(r_i)}{t_{k+1}\omega(\frac{1}{8}r_{k+1})}$$
$$\le \gamma_1(n) + \gamma_2(n)\frac{\omega(r_{k+1})}{\omega(\frac{1}{8}r_{k+1})}\ \le \gamma_3(n).$$

Thus (iii) and the lemma are proved. □

Note now that by (4.4), if $y \in X_{k+1}$, it belongs either to $\tilde{X}_{k+1}$ or to $2r_k\vec{e}_1 - \tilde{X}_{k+1}$. Then Lemmas 4.4 and 4.5 imply that $f_{k+1,y}$ exists for every $y \in X_{k+1}$. The proof of Lemma 4.2 is finished. □

Lemma 4.6. *Let $f \in C^{1,\omega}(\mathbb{R}^n)$ satisfy*

$$f(x) = 0 \text{ if } x \in X_k\backslash\{b^{(k)}\}\ (1 \le k \le n).$$

Then there is a constant $\gamma = \gamma(n)$ such that

$$|f(b^{(k)})| \le \gamma |f|_{C^{1,\omega}} \left\{\sum_{i=1}^{k} t_i\omega(r_{i-1})\right\}. \tag{4.16}$$

Here

$$|f|_{C^{1,\omega}}\ :=\ \sup_{x\ne y}\ \max_{i=1,\ldots,n}\ \frac{|\frac{\partial f}{\partial x_i}(x) - \frac{\partial f}{\partial x_i}(y)|}{\omega(\|x-y\|)}$$

stands for the "homogeneous" part of $C^{1,\omega}$-norm.

Proof (by induction on k). Let $k=1$, i.e., $X_1 = \{0, t_1\vec{e}_1, 2t_1\vec{e}_1\}$ and $b^{(1)} = t_1\vec{e}_1$. Since $f \in C^{1,\omega}(\mathbb{R}^n)$, $f(0) = f(2t_1\vec{e}_1) = 0$ and $t_1 = r_0$, we have

$$|f(b^{(1)})| = \left|\int_0^{t_1}\left\{\frac{\partial f}{\partial x_1}(u + t_1\vec{e}_1) - \frac{\partial f}{\partial x_1}(u)\right\}du\right|$$
$$\le \int_0^{t_1}\left|\frac{\partial f}{\partial x_1}(u + t_1\vec{e}_1) - \frac{\partial f}{\partial x_1}(u)\right|du \le t_1|f|_{C^{1,\omega}}\omega(r_0).$$

Thus the lemma is proved for the case $k=1$.

Assume now that (4.16) holds for $k > 1$. Let $f \in C^{1,\omega}(\mathbb{R}^n)$ and $f(x) = 0$ for all $x \in X_{k+1}\backslash\{b^{(k+1)}\}$. Define $f_1 : \mathbb{R}^n \to \mathbb{R}$ by

$$f_1(x) :=\ f(2r_{k-1}\vec{e}_1 - x).$$

Then f_1 clearly equals 0 on $X_k\backslash\{b_k\}$. Applying the induction assumption to f_1 we get

$$|f(a^{(k)})| = |f_1(b^{(k)})| \le \gamma(k)|f_1|_{C^{1,\omega}}\left\{\sum_{i=1}^{k} t_i\omega(r_{i-1})\right\} \tag{4.17}$$
$$= \gamma(k)|f|_{C^{1,\omega}}\left\{\sum_{i=1}^{k} t_i\omega(r_{i-1})\right\}.$$

Define now $f_2 : \mathbb{R}^n \to \mathbb{R}$ by

$$f_2(x) := f(2(r_k - r_{k-1})\vec{e}_1 + x).$$

Clearly, f_2 is equal to 0 on $X_k\backslash\{b^{(k)}\}$ so that by the induction assumption applied to f_2 one gets

$$|f_2(b^{(k)})| \le \gamma(k)|f_2|_{C^{1,\omega}} \sum_{i=1}^{k} t_i\omega(r_{i-1}) = \gamma(k)|f|_{C^{1,\omega}} \left\{\sum_{i=1}^{k} t_i\omega(r_{i-1})\right\}.$$

Set $\hat{a} := 2r_k\vec{e}_1 - a^{(k)}$. Then by (4.2) we have $\hat{a} = a^{(k+1)} + t_{k+1}\vec{e}_{k+1}$ and

$$f(\hat{a}) = f(2r_k\vec{e}_1 - a^{(k)}) = f(2r_k\vec{e}_1 - 2r_{k-1}\vec{e}_1 + b^{(k)}) = f_2(b^{(k)}).$$

Hence

$$|f(\hat{a})| \le \gamma(k)|f|_{C^{1,\omega}} \left\{\sum_{i=1}^{k} t_i\omega(r_{i-1})\right\}. \tag{4.18}$$

Inequalities (4.17) and (4.18), and the equality $f(a^{(k+1)}) = 0$ lead to the following estimate for $|f(b^{(k+1)})|$

$$\begin{aligned} |f(b^{(k+1)})| &= |f(b^{(k+1)}) - f(a^{(k)}) - (f(\hat{a}) - f(a^{(k+1)})) + f(a^{(k)}) + f(\hat{a})| \\ &\le |\int_0^{t_{k+1}} \left(\frac{\partial f}{\partial x_{k+1}}(a^{(k)} + t\vec{e}_{k+1}) - \frac{\partial f}{\partial x_{k+1}}(a^{(k+1)} + t\vec{e}_{k+1})\right) dt| \\ &\quad + |f(a^{(k)})| + |f(\hat{a})| \\ &\le |f|_{C^{1,\omega}} \left\{ t_{k+1}\omega(\|a^{(k)} - a^{(k+1)}\|) + 2\gamma(k)\left(\sum_{i=1}^{k} t_i\omega(r_{i-1})\right)\right\}. \end{aligned}$$

But by (4.2) $\|a^{(k)}\| \le 2r_{k-1}$ $(k = 1, 2, \dots, n)$ and therefore, the above inequality implies

$$\begin{aligned} |f(b^{(k+1)})| &\le 2\gamma(k)|f|_{C^{1,\omega}} \left\{ t_{k+1}(\omega(\|a^{(k)}\|) + \omega(\|a^{(k+1)}\|)) + \sum_{i=1}^{k} t_i\omega(r_{i-1})\right\} \\ &\le 2\gamma(k)|f|_{C^{1,\omega}} \left\{ 2t_{k+1}\omega(2r_k) + \sum_{i=1}^{k} t_i\omega(r_{i-1})\right\} \\ &\le 8\gamma(k)|f|_{C^{1,\omega}} \left(\sum_{i=1}^{k+1} t_i\omega(r_{i-1})\right). \end{aligned}$$

which completes the proof of the lemma. □

Now apply Lemma 4.7 to the above defined function $f_n : X_n \to \mathbb{R}$, see (4.6), and for $k = n$. Since each extension of f_n to a function $\tilde{f} \in C^{1,\omega}(\mathbb{R}^n)$ satisfies the conditions of the lemma and $\tilde{f}(b^{(n)}) = f_n(b^{(n)}) = \min\{t_i\omega(r_i) : i = 1, \dots, n\}$, we get the following

Corollary 4.7.

$$\|f_n\|_{C^{1,\omega}|_{X_n}} \ \ge\ \gamma\ \frac{\min\limits_{k=1,\dots,n} t_k\omega(r_k)}{\sum\limits_{k=1}^{n} t_k\omega(r_{k-1})}$$

with $\gamma > 0$ depending only on n.

Now we define the required set $X = X_\omega$ by

$$X := \bigcup_{m=0}^{\infty} X^{(m)} \tag{4.19}$$

and the function $F = F_\omega : X \to \mathbb{R}$ by

$$F|_{X^{(m)}} := F^{(m)} \quad (m = 0, 1, \dots).$$

Since $X^{(m)} \cap X^{(m')} \neq \emptyset$ for $m \neq m'$, this definition is correct. To complete the proof of Proposition 4.1 one first defines for every $m = 1, 2, \dots$ special number sequences $\left\{r_k^{(m)}\right\}_{k=0}^{n}$ and $\left\{t_k^{(m)}\right\}_{k=1}^{n}$ satisfying condition (4.1). Using these sequences one defines the sequences of sets $\left\{X_k^{(m)}\right\}_{k=1}^{n}$ and the functions $\left\{f_k^{(m)}\right\}_{k=1}^{n}$ by formulas (4.2)-(4.6). Finally, one puts for $m = 1, 2, \dots$

$$X^{(m)} := X_n^{(m)} + r_n^{(m)}\vec{e}_1,\ F^{(m)}(x) := f_n^{(m)}(x - r_n^{(m)}\vec{e}_1),\ x \in X^{(m)}. \tag{4.20}$$

We also set $X^{(0)} := \{0\}$ and $F^{(0)}(x) := 0,\ x \in X^{(0)}\ (= \{0\})$.

The sequences $\left\{r_k^{(m)}\right\}_{k=1}^{n}$ and $\left\{t_k^{(m)}\right\}_{k=1}^{n}$ $(m = 1, 2, \dots)$ are introduced as follows. Put $r_n^{(m)} := 4^{-m}$ and define the number $r_k^{(m)}$ for $0 \le k \le n-1$ as a root of the equation

$$\omega(x) \ = \ \frac{\omega(4^{-m})}{2^{(m+3)(n-k)}}\ .$$

Since ω is continuous and non-decreasing, the equation has a solution. By definition this sequence satisfies

$$\frac{\omega(r_k^{(m)})}{\omega(r_{k-1}^{(m)})} \ = \ 2^{m+3}\ (k = 1, 2, \dots, n). \tag{4.21}$$

Since $t \to \omega(t)/t$ is non-increasing, one has also

$$\frac{r_{k+1}^{(m)}}{r_k^{(m)}} \ \ge \ \frac{\omega\left(r_{k+1}^{(m)}\right)}{\omega\left(r_k^{(m)}\right)} \ = \ 2^{m+3} \ \ge \ 8.$$

One now defines the sequence $\left\{t_k^{(m)}\right\}_{k=1}^{n}$ by

$$t_1^{(m)} := r_0^{(m)} \ \text{ and } \ t_k^{(m)} := r_0^{(m)}\ \frac{\omega(r_1^{(m)})}{\omega(r_k^{(m)})},\ k = 2, \dots, n. \tag{4.22}$$

These two sequences are readily seen to be monotone and to satisfy (4.1).

Having disposed of this preliminary step, we can now establish the required properties of the function $F = F_\omega$.

Lemma 4.8. *The restriction $F|_Y$ of F to an arbitrary subset $Y \subset X = X_\omega$ with* $\text{card}(Y) \le 3 \cdot 2^{n-1} - 1$ *has an extension to a function $F_Y \in C^{1,\omega}(\mathbb{R}^n)$ satisfying* $\|F_Y\|_{C^{1,\omega}(\mathbb{R}^n)} \le \gamma(n)$.

Proof. Given $Y \subset X$ with $\text{card}(Y) \le 3 \cdot 2^{n-1} - 1$ let $Y^{(m)} := Y \cap X^{(m)}$. Applying Lemma 4.2, (4.20) and (4.22) to the function $f_n^{(m)} : X_n^{(m)} \to \mathbb{R}$ one finds a function $F_{Y,m} \in C^{1,\omega}(\mathbb{R}^n)$ satisfying

(a) $F_{Y,m}$ coincides with $F^{(m)}$ on $Y^{(m)}$, $m = 1, 2, \dots$;

(b) $\operatorname{supp}(F_{Y,m}) \subset H_m := \{x \in \mathbb{R}^n : \ |x_1 - 2r_n^{(m)}| \le \frac{1}{2} r_n^{(m)}\}$;
(c) $\|F_{Y,m}\|_{C^{1,\omega}(\mathbb{R}^n)} \le \gamma_1(n)$.

Define now the required extension F_Y by

$$F_Y(x) := \begin{cases} F_{Y,m}(x), & \text{if } x \in H_m \text{ for some } m, \\ 0, & \text{if } x \notin \bigcup\limits_{m=1}^{\infty} H_m. \end{cases} \tag{4.23}$$

By (4.1)–(4.4) and (4.20), $X^{(m)} \subset H_m$ and, moreover, $\|x - y\| \ge \frac{1}{2} r_n^{(m)}$ for every $x \in H_m$ and $y \in H_\ell$ with $\ell \ge m$. From this, (4.19), (4.20) and condition (a), it follows that F_Y interpolates F on Y. In addition, by conditions (b), (c) and (4.23), the function F_Y is differentiable and, moreover,

$$\nabla F_Y(x) = \begin{cases} \nabla F_{Y,m}, & \text{if } x \in H_m \text{ for some } m, \\ 0, & \text{if } x \notin \bigcup\limits_{m=1}^{\infty} H_m. \end{cases}$$

Then (4.23) and (c) immediately lead to the inequalities

$$\sup_{\mathbb{R}^n} |F_Y| + \sup_{\mathbb{R}^n} \|\nabla F\| = \sup_m \{\sup_{\mathbb{R}^n} |F_{Y,m}|\} + \sup_m \{\sup_{\mathbb{R}^n} \|\nabla F_{Y,m}\|\} \le 2\gamma_1(n),$$

$$\|\nabla F_Y(x) - \nabla F_Y(y)\| \le 2\gamma_1(n)\omega(\|x - y\|), \ (x, y \in \mathbb{R}^n).$$

Hence $\|F_Y\|_{C^{1,\omega}(\mathbb{R}^n)} \le 4\gamma_1(n)$ and the proof is complete. □

Thus to complete the proof of Proposition 4.1 it remains to establish

Lemma 4.9. *$F \notin C^{1,\omega}(\mathbb{R}^n)|_X$.*

Proof. Suppose that, on the contrary, $F \in C^{1,\omega}(\mathbb{R}^n)|_X$. Then, by (4.19)–(4.20)

$$+\infty > \|F\|_{C^{1,\omega}(\mathbb{R}^n)|_X} \ \ge \sup_m \|F^{(m)}\|_{C^{1,\omega}(\mathbb{R}^n)|_{X^{(m)}}} = \sup_m \|f_n^{(m)}\|_{C^{1,\omega}(\mathbb{R}^n)|_{X_n^{(m)}}}.$$

To estimate from below the latter supremum, one applies Corollary 4.7. Then we get

$$\|f_n^{(m)}\|_{C^{1,\omega}(\mathbb{R}^n)|_{X_n^{(m)}}} \ge \gamma \ \frac{\min\limits_{k=1,\dots,n} t_k^{(m)} \omega(r_k^{(m)})}{\sum\limits_{k=1}^{n} t_k^{(m)} \omega(r_{k-1}^{(m)})},$$

where $\gamma = \gamma(n) > 0$ depends on n only.

In turn, by (4.22),

$$\min_{k=1,\dots,n} t_k^{(m)} \omega(r_k^{(m)}) = t_1^{(m)} \omega(r_1^{(m)})$$

and by (4.21) and (4.22)

$$\sum_{k=1}^{n} t_k^{(m)} \omega(r_{k-1}^{(m)}) = 2^{-m-3} n t_1^{(m)} \omega(r_1^{(m)}).$$

Hence

$$\|F\|_{C^{1,\omega}(\mathbb{R}^n)|_X} \ge \sup_m \|f_n^{(m)}\|_{C^{1,\omega}(\mathbb{R}^n)|_{X_n^{(m)}}} \ge \sup_m \left\{ \frac{1}{n} \gamma 2^{m+3} \right\} = +\infty,$$

a contradiction.

Thus Proposition 4.1 and Theorem 1.3 are completely proved. □

5. Concluding remarks

1. Let us note a consequence of Theorem 1.3 related to description of the trace spaces $C^1(\mathbb{R}^n)|_X$. For this goal we modify Definition 1.2 replacing the condition of boundedness of the set $\{f_Y : Y \subset X, \mathrm{card}(Y) \le N\}$ by that of *precompactness* (in A). We denote the corresponding constant by $\tilde{N}[A]$. Then the aforementioned consequence of Theorem 1.3 states the following.

Theorem 5.1. $\tilde{N}[C^1(\mathbb{R}^n)] \le 3 \cdot 2^{n-1}$.

2. From some results of Glaeser's paper [G] (see, in particular, Theorem 1 in Section II.5 and Proposition 7 in Section II.6 there) one can deduce the following interesting characterization of $C^1(\mathbb{R}^n)|_X$.

Let $f : X \to \mathbb{R}$ be a *continuous* function and $\Gamma_f := \{(x, f(x)) \in \mathbb{R}^{n+1} : x \in X\}$ be its graph. Define $D_0(x; f)$ to be the set of limits of straight lines MM' passing through pairs M, M' of points in Γ_f when M, M' tend independently to $(x, f(x))$. Further, define the set $D_1(x; f)$, $x \in X$, by the extension of $D_0(x; f)$ in the following way. In the first step we replace $D_0(x; f)$ by the set $D_{\frac{1}{2}}(x; f)$ of all straight lines from the affine hull of $D_0(x; f)$ passing through $(x, f(x))$. In the second step we determine $D_1(x; f)$ as the set of all straight lines $l = \lim_{i\to\infty} l_i$ where $l_i \in D_{\frac{1}{2}}(x_i; f)$ and $x_i \to x$, $x_i \in X$. Continuing this procedure we subsequently obtain $D_2(x; f)$, $D_3(x; f)$, etc.

Theorem 5.2. $f \in C^1(\mathbb{R}^n)|_X$ *if and only if* $\dim D_{2n}(x; f) \le n$.

3. It is worth noting another important problem also originated in the Whitney papers [W1], [W2]. It concerns the existence of a *linear* bounded extension operator from $A|_X$ into A (in the notations of Definition 1.2). It was proved in [BS2] and [BS3] that the problem has a positive answer for the space $C^{1,\omega}(\mathbb{R}^n)$ and the jet-space $J^k\Lambda^\omega(\mathbb{R}^n)$. Here $\Lambda^\omega(\mathbb{R}^n)$ is a (generalized) Zygmund space; see, e.g., [St], Ch. 5, for its definition.

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